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Numerical solution of some ordinary differential equations occurring in plate deflection theory

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SUMMARY

A certain fourth-order differential equation is solved numerically by the method of finite differences. Conditions on the original differential equation are given which are sufficient to quarantee that the matrix thus produced is monotone so that a straightforward error analysis is possible. This error analysis is given in detail. Examples are given which demonstrate the validity of this error analysis.

1. Introduction

We consider the problem of bending of a uniformly loaded rectangular plate of length l supported over the entire surface by an elastic foundation and rigidly supported along the edges. The deflection w of the plate satisfies the differential system

$$\begin{aligned} &D\frac{d^4 w}{dx^4} + kw = q\,, \\ &w(\pm l/2) = 0\,, \quad \frac{d^2 w}{dx^2}(\pm l/2) = 0\,, \end{aligned} \tag{1.1}$$

where D is the flexural rigidity of the plate, k is the reaction of the foundation per unit area for a deflection equal to unity, and q denotes the intensity of the continuously distributed load. The details of the mechanical interpretation of (1.1) and its analytic solution for the special case where D, k, q are constants are given in [4, p. 30]. Mathematically (1.1) belongs to a general class of boundary value problems of the form

$$\begin{aligned} &\frac{d^4 y}{dx^4} + f(x)y = g(x)\,, \qquad f(x) \geqq 0\,, \\ &y(a) = \alpha_1\,,\ \ y(b) = \alpha_2\,,\ \ y''(a) = \beta_1\,,\ \ y''(b) = \beta_2\,. \end{aligned} \tag{1.2}$$

The analytical solution of the system (1.2) for all $f(x)$ and $g(x)$ cannot be found.

Faced with this difficulty we resort to numerical methods for obtaining an approximate solution of the system. The most widely used technique for approximating y over a finite set of grid points $\{x_n\} \subset (a, b)$ is by finite difference methods. Many authors have considered the use of finite difference methods for solving ordinary differential systems, see for instance [1, 2, 5].

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Our purpose in this paper is to formulate a second order finite difference method for the approximate determination of y over $\{x_n\}$ and to study the error in this method.

We formally introduce the method in the next section. In Section 3 we begin an analysis of the error incurred by using the method. Since completion of this analysis requires that a certain set of matrices be monotone, we denote Section 4 to a proof of this fact. The convergence of our finite difference method is proved in Section 5. In conclusion, a survey of some experimental results is given in Section 6 to demonstrate the practical usefulness of the method.

2. Finite difference scheme

For a direct numerical solution of the boundary value problem (1.2) we first introduce a finite sequence $\{x_n\}$ so that

$$x_n = a + nh\,, \qquad n = 0, 1, \ldots, N+1\,, \tag{2.1}$$

where

$$x_0 = a\,, \quad x_{N+1} = b\,,$$

and

$$h = (b-a)/(N+1)\,. \tag{2.2}$$

We let y_i be the approximation to the exact solution of the system (1.2) at $x = x_i$, namely $y(x_i)$.

The vectors $Y = (y_i)$ and $\tilde{Y} = (y(x_i))$ will satisfy the matrix equations

$$AY = C \text{ and } A\tilde{Y} = C + T\,, \tag{2.3}$$

where $A = (a_{ij})$ is a five-band matrix of order N with

$$a_{ij} = \begin{cases} 5 + h^4 f_i\,, & i = j = 1, N \\ 6 + h^4 f_i\,, & i = j = 2, 3, \ldots, N-1 \\ -4\,, & |i-j| = 1 \\ 1\,, & |i-j| = 2 \\ 0\,, & |i-j| > 2 \end{cases} \tag{2.4}$$

and $C = (c_i)$ is the N-dimensional column vector given by

$$\begin{aligned} c_1 &= h^4 g_1 + 2\alpha_1 - h^2 \beta_1 + \frac{h^4}{12}(f_0 \alpha_1 - g_0)\,, \\ c_2 &= h^4 g_2 - \alpha_1\,, \\ c_i &= h^4 g_i\,, \qquad i = 3, 4, \ldots, N-2\,, \\ c_{N-1} &= h^4 g_{N-1} - \alpha_2\,, \\ c_N &= h^4 g_N + 2\alpha_2 - h^2 \beta_2 + \frac{h^4}{12}(f_{N+1}\alpha_2 - g_{N+1})\,. \end{aligned} \tag{2.5}$$

Thus, the left equation in (2.3) is

$$\begin{aligned} &\text{(i)} \quad (5 + h^4 f_1) y_1 - 4y_2 + y_3 = c_1 \\ &\text{(ii)} \quad y_{n-2} - 4y_{n-1} + (6 + h^4 f_n) y_n - 4y_{n+1} + y_{n+2} = h^4 g_n\,, \qquad n = 2, 3, \ldots, N-1 \\ &\text{(iii)} \quad y_{N-2} - 4y_{N-1} + (5 + h^4 f_N) y_N = c_N \end{aligned} \tag{2.6}$$

with, of course,

$$y_0 = y(x_0) = \alpha_1 \text{ and } y_{N+1} = y(x_{N+1}) = \alpha_2\,. \tag{2.7}$$

3. Truncation error, error equation

The right equation in (2.3) serves to define the local truncation error vector $T=(t_i)$ associated with (2.6), the finite difference approximation to the differential system (1.2). By using the relations

$$y''(x_0)=\beta_1\,,\quad y''(x_{N+1})=\beta_2\,,\quad y^{(4)}(x_i)=-f_i y(x_i)+g_i$$

and (2.7), we may eliminate α_i, β_i, f_i, and g_i to produce

$$\begin{aligned}
&\text{(i)}\ \ t_1=-2y(x_0)+5y(x_1)-4y(x_2)+y(x_3)\\
&\qquad\quad -h^4y^{(4)}(x_1)+h^2y''(x_0)+\tfrac{1}{12}h^4y^{(4)}(x_0)\\
&\text{(ii)}\ \ t_n=y(x_{n-2})-4y(x_{n-1})+6y(x_n)-4y(x_{n+1})+y(x_{n+2})\\
&\qquad\quad -h^4y^{(4)}(x_n)\,,\qquad n=2,3,\ldots,N-1\\
&\text{(iii)}\ t_N=-2y(x_{N+1})+5y(x_N)-4y(x_{N-1})+y(x_{N-2})\\
&\qquad\quad -h^4y^{(4)}(x_N)+h^2y''(x_{N+1})+\tfrac{1}{12}h^4y^{(4)}(x_{N+1})\,.
\end{aligned}\tag{3.1}$$

Using Taylor's formula with integral representation of the remainder, namely

$$y(x_{n+j})=\sum_{i=0}^{k}\frac{(jh)^i}{i!}\,y^{(i)}(x_n)+\frac{(jh)^{k+1}}{k!}\int_0^1(1-t)^k y^{(k+1)}(x_n+jht)dt\,,$$

repeatedly in (3.1) yields, after some variable changes of the form $s=\pm jt$,

$$\begin{aligned}
&\text{(i)}\ \ t_1=h^6\int_0^3 G_1(s)\,y^{(6)}(x_0+hs)ds\\
&\text{(ii)}\ \ t_n=h^6\int_{-2}^{2} G_2(s)\,y^{(6)}(x_n+hs)ds\,,\qquad n=2,3,\ldots,N-1\\
&\text{(iii)}\ t_N=h^6\int_0^3 G_1(s)\,y^{(6)}\ (x_{N+1}-hs)ds
\end{aligned}\tag{3.2}$$

where the kernels $G_i(s)$ are given by

$$\begin{aligned}
G_1(s)&=\tfrac{1}{120}(3-s)^5\,, && 2\leqq s\leqq 3\\
&=\tfrac{1}{120}(3-s)^5-\tfrac{1}{30}(2-s)^5\,, && 1\leqq s\leqq 2\\
&=\tfrac{1}{120}(3-s)^5-\tfrac{1}{30}(2-s)^5+\tfrac{1}{24}(1-s)^5-(1-s)\,, && 0\leqq s\leqq 1
\end{aligned}$$

and

$$\begin{aligned}
G_2(s)&=G_2(-s)=\tfrac{1}{120}(2-s)^5\,, && 1\leqq s\leqq 2\\
&=\tfrac{1}{120}(2-s)^5-\tfrac{1}{30}(1-s)^5\,, && 0\leqq s\leqq 1\,.
\end{aligned}$$

It is easily established that $G_i(s)\geqq 0$ for $i=1,2$ so that by the second law of the mean

$$\begin{aligned}
&\text{(i)}\ \ t_1=\tfrac{59}{360}h^6y^6(\xi_1)\,,\quad x_0<\xi_1<x_3\\
&\text{(ii)}\ \ t_n=\tfrac{1}{6}h^6y^{(6)}(\xi_n)\,,\quad x_{n-2}<\xi_n<x_{n+2}\,,\quad n=2,3,\ldots,N-1\\
&\text{(iii)}\ t_N=\tfrac{59}{360}h^6y^{(6)}(\xi_N)\,,\ x_{N-2}<\xi_N<x_{N+1}\,.
\end{aligned}\tag{3.3}$$

Subtracting the equations of (2.3) yields

$$AE=T\,,\tag{3.4}$$

where $E=(e_i)$ is the error vector with e_i the error of discretization defined by

$$e_i=y(x_i)-y_i\,.\tag{3.5}$$

Thus, e_i is the amount by which the numerical approximation y_i deviates from the actual solution $y(x_i)$ of (1.2) at $x=x_i$.

From (3.3) and (3.4), the jth entry of AE satisfies

$$-\frac{h^6}{6} M_6 \leqq (AE)_j = t_j \leqq \frac{h^6}{6} M_6 , \tag{3.6}$$

where $M_6 = \max |y^{(6)}(x)|$.

Any further analysis now depends entirely on the properties of the matrices A and A_0, where A_0 is the five-band matrix obtained from A by setting $f_i \equiv 0$, so that

$$A_0 = \begin{bmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{bmatrix}$$

4. Properties of the matrix A_0 and A

We shall prove that A_0 and, under certain conditions, A are both monotone matrices (that is the elements of A_0^{-1} and A^{-1} are non-negative).

We shall first deal with matrix A_0. Let P be a tridiagonal matrix given by

$$P = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \cdots\cdots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \tag{4.1}$$

and if $P^{-1} = (P_{ij})$, then

$$P_{ij} = \begin{cases} j(N-i+1)/(N+1) > 0 , & i \geqq j \\ i(N-j+1)/(N+1) > 0 , & i \leqq j , \end{cases} \tag{4.2}$$

see Henrici [2, p. 363]. The matrix P is a monotone matrix. Furthermore, it can be verified that

$$A_0 = P^2 , \tag{4.3}$$

and hence

$$A_0^{-1} = [P^{-1}]^2 . \tag{4.4}$$

From (4.4) it follows that A_0 is also a monotone matrix. We can determine the elements of A_0^{-1} using (4.4). If

$$A_0^{-1} = (a_{ij}^*) , \tag{4.5}$$

then the element a_{ij}^* is obtained by multiplying the ith row of P^{-1}, namely,

$$\frac{1}{N+1} [1 \cdot (N-i+1) ,\ 2(N-i+1) , \ldots, i(N-i+1) , i\,(N-i-1) , \ldots, i \cdot 2,\ i \cdot 1] \tag{4.6}$$

by the jth column of P^{-1}, namely,

$$\frac{1}{N+1} [1 \cdot (N-j+1) ,\ 2(N-j+1) , \ldots, j(N-j+1) ,\ j(N-j) ,\\ j\,(N-j-1) , \ldots, j \cdot 2 ,\ j \cdot 1]^t , \tag{4.7}$$

where t denotes the operation of transposition. For the determination of a^*_{ij} explicitly, we will need the following algebraic identities.

$$\text{(i)}\quad \sum_{K=1}^{j} K^2 = \tfrac{1}{6}j(j+1)(2j+1)$$

$$\text{(ii)}\quad \sum_{K=j+1}^{i} K(N-K+1) = \tfrac{1}{6}(i-j)[3N(i+j+1)-2(i^2+j^2+ij-1)] \tag{4.8}$$

$$\text{(iii)}\quad \sum_{K=i+1}^{N} (N-K+1)^2 = \tfrac{1}{6}(N-i)(N-i+1)(2N-2i+1)\,.$$

We note that 4.8 (i) is a well-known identity and 4.8 (ii) and 4.8 (iii) can be proved by easy algebraic manipulations using 4.8 (i).

Now assume $i \geqq j$, then

$$a^*_{ij} = \sum_{K=1}^{N} P_{iK}P_{Kj}$$
$$= \sum_{K=1}^{j} P_{iK}P_{Kj} + \sum_{K=j+1}^{i} P_{iK}P_{Kj} + \sum_{K=i+1}^{N} P_{iK}P_{Kj}\,,$$

and on using (4.6) and (4.7) we obtain

$$(N+1)^2 a^*_{ij} = \sum_{K=1}^{j} K(N-i+1)K(N-j+1) + \sum_{K=j+1}^{i} K(N-i+1)j(N-K+1) + \sum_{K=i+1}^{N} i(N-K+1)j(N-K+1)$$
$$= (N-i+1)(N-j+1)\sum_{K=1}^{j} K^2 + j(N-i+1)\sum_{K=j+1}^{i} K(N-K+1) + ij\sum_{K=i+1}^{N}(N-K+1)^2\,,$$

or

$$a^*_{ij} = \tfrac{1}{6}j(N-i+1)\left[2i + \frac{1}{N+1} - \frac{i^2+j^2}{N+1}\right] > 0\,, \text{ using } (4.8)\,. \tag{4.9}$$

Since A_0^{-1} is symmetric, hence on interchanging i and j in (4.9), we obtain a^*_{ij} for $i \leqq j$. Thus we have proved the following theorem.

Theorem 4.1. *The real, symmetric matrix A_0 defined by* (4.3) *is a monotone matrix and if $A_0^{-1} = (a^*_{ij})$, then A_0^{-1} is symmetric and*

$$a^*_{ij} = \begin{cases} \tfrac{1}{6}j(N-i+1)\left[2i + \dfrac{1}{N+1} - \dfrac{i^2+j^2}{N+1}\right] > 0\,, & i \geqq j \\ \tfrac{1}{6}i(N-j+1)\left[2j + \dfrac{1}{N+1} - \dfrac{i^2+j^2}{N+1}\right] > 0\,, & i \leqq j\,. \end{cases}$$

We will now proceed to prove that the matrix A given by (2.4) is a monotone matrix under certain conditions. To this end we define $d_n = h^2 f_n^{\frac{1}{2}}$ and set $D = \mathrm{diag}(d_n)$. Note that $D^2 = A - A_0$.

Lemma 4.1. *The real symmetric matrix $P^{-1} - DP^{-2}$ has nonnegative entries if and only if*

$$d_i \leqq 6/(N+1-i)(N+1+i) \quad \text{for} \quad 1 \leqq i \leqq (N+1)/2$$
$$\leqq 6/i(2N+2-i) \qquad \text{for} \quad (N+1)/2 \leqq i \leqq N\,. \tag{4.10}$$

Proof: This lemma follows from (4.2) and Theorem 4.1 when we examine the (i, j) entry of $P^{-1}-DP^{-2}$. For a fixed i and $j \leqq i$, this entry is

$$[j(N+i-1)/6(N+1)][6-d_i(2iN+2i+1-i^2-j^2)]$$

while, for $j \geqq i$, this entry is

$$[i(N+j-1)/6(N+1)][6-d_i(2jN+2j+1-i^2-j^2)] .$$

In the former case, a necessary and sufficient condition is that

$$d_i \leqq 6/(2iN+2i+1-i^2-j^2) \text{ for } j=1, \ldots, i$$

or, equivalently,

$$d_i \leqq 6/i(2N+2-i) . \tag{4.11}$$

Similarly,

$$d_i \leqq 6/(2jN+2j+1-i^2-j^2) \text{ for } j=i, \ldots, N$$

or, equivalently,

$$d_i \leqq 6/(N+1-i)(N+1+i) . \tag{4.12}$$

A comparison of (4.11) and (4.12) then yields the lemma.

Lemma 4.2. *A sufficient condition that $P^{-1}-DP^{-2}$ have nonnegative entries (for all N) is that*

$$0 \leqq f(x) \leqq 36/(b-x)^2(b+x-2a)^2 \text{ for } a \leqq x \leqq (a+b)/2 \tag{4.13}$$

and

$$0 \leqq f(x) \leqq 36/(x-a)^2(2b-x-a)^2 \text{ for } (a+b)/2 \leqq x \leqq b \tag{4.14}$$

Proof: From (2.1) and (2.2) we may square both members of (4.10) to obtain

$$h^4 f_i \leqq 36h^4/(b-x_i)^2(b+x_i-2a)^2 \text{ for } 1 \leqq i \leqq (N+1)/2$$

and

$$h^4 f_i \leqq 36h^4/(x_i-a)^2(2b-x_i-a)^2 \text{ for } (N+1)/2 \leqq i \leqq N .$$

Since $f_i = f(x_i)$, these relations are implied by the condition of the lemma. The proof is complete.
Combining Lemma 4.2 and the Neumann series

$$(I+M)^{-1} = I-M+M^2-M^3+\ldots ,$$

which is valid when $\rho(M)$, the spectral radius of M, is less than 1, produces

Theorem 4.2. *If*

$$0 \leqq f(x) \leqq 36/(b-a)^4 , \tag{4.15}$$

then the matrix A given by (2.4) is monotone.

Proof: From the preceding discussion,

$$A = A_0 + D^2 = P^2 + D^2 .$$

Hence, $AP^{-2} = I + D^2P^{-2}$ so that

$$\begin{aligned} P^2A^{-1} &= (I+D^2P^{-2})^{-1} \\ &= I-(D^2P^{-2})+(D^2P^{-2})^2-(D^2P^{-2})^3+\ldots \\ &= [I-D^2P^{-2}][I+(D^2P^{-2})^2+(D^2P^{-2})^4+\ldots] \end{aligned}$$

if the two infinite series converge.

In a manner similar to the discussion in [5, p. 202] one can establish that the eigenvalues of P are

$$\lambda_i = 4\sin^2\left(\frac{i\pi}{2N+2}\right), \qquad i=1,2,\ldots,N.$$

Hence, since $x\sin(1/x)$ is an increasing function of x for $x \geqq (2N+2)/\pi$,

$$\rho(P^{-1}) = \tfrac{1}{4}\csc^2\left(\frac{\pi}{2N+2}\right) < \frac{(N+1)^2}{8}$$

and, by [5, p. 47, Th. 2.8] and (4.15)

$$\rho(D^2P^{-2}) \leqq (\max d_i)^2 \rho^2(P^{-1}) < 9/16.$$

Therefore, the two infinite series converge.

Let $\tilde{D}$ be formed from D by replacing each diagonal entry by $\max d_i$. Then

$$\begin{aligned} A^{-1} &= [P^{-2} - P^{-2}D^2P^2][I + (D^2P^{-2})^2 + \ldots] \\ &\geqq [P^{-2} - \tilde{D}^2P^{-4}][I + (D^2P^{-2})^2 + \ldots], \end{aligned}$$

where the inequality is valid because all factors involve nonnegative terms.

Hence, to show that A is monotone, it suffices to show that

$$P^{-2} - \tilde{D}^2P^{-4} \geqq 0. \tag{4.16}$$

Since $P^{-2} - \tilde{D}^2P^{-4} = (P^{-1} - \tilde{D}P^{-2})(P^{-1} + \tilde{D}P^{-2})$, (4.16) will be implied by

$$P^{-1} - \tilde{D}P^{-2} \geqq 0. \tag{4.17}$$

Now, (4.15) implies (4.13) and (4.14) with $f(x)$ replaced by $\max f(x)$. Hence, Lemma 4.2 is valid with D replaced by $\tilde{D}$. Thus, (4.17) holds and Theorem 4.2 is proved.

Note that (4.16) is a weaker condition than (4.17). An improved theorem may follow from (4.16) and the explicit entries of P^{-4}.

Remark: While stronger theorems no doubt exist, the upper bound in (4.15) cannot be disregarded altogether. For example, if, in (2.4), each $f_i = 1000/(b-a)^4$, then it can be shown that for all $N>5$, the $i=1$, $j=N$ entry of A^{-1} is negative.

Having proved that both matrices A and A_0 are monotone satisfying $A \geqq A_0$, it therefore follows from [2, p. 362, Th. 7.5] that

$$0 < A^{-1} < A_0^{-1}. \tag{4.18}$$

Combining (3.6) and (4.18) will provide the next step in the error analysis.

5. Analysis of discretization error, convergence of the method

Our main concern here is to derive a bound on $|e_i|$, defined by (3.5), and a bound on $\|E\| = \max|e_i|$. In order to do so we will need the following lemmas.

Lemma 5.1. *For* $i=1,2,\ldots,N$

$$\sum_{j=1}^{i} a_{ij}^* = \frac{i(N+1-i)}{24(N+1)}[(N+1)(4i^2+4i) - (3i^3+4i^2-i-2)]$$

and

$$\begin{aligned} \sum_{j=i+1}^{N} a_{ij}^* = \frac{i(N+1-i)}{24(N+1)}[&(N+1)^3 + (N+1)^2 i - (N+1)(5i^2+4i-1) \\ &+ (3i^3+4i^2-i-2)]. \end{aligned}$$

We can easily prove Lemma 5.1 by the application of Theorem 4.1, although for the second sum we need the following identities

$$\sum_{j=i+1}^{N} j = \tfrac{1}{2}(N+1+i)(N-i),$$

$$\sum_{j=i+1}^{N} j^2 = \tfrac{1}{6}[N(N+1)(2N+1)-i(i+1)(2i+1)],$$

$$\sum_{j=i+1}^{N} j^3 = \tfrac{1}{4}[N^2(N+1)^2-i^2(i+1)^2].$$

Lemma 5.2. *For* $i=1, 2, \ldots, N$

$$\sum_{j=1}^{N} a_{ij}^* = \frac{i(N+1-i)}{24}[i(N+1-i)+(N+1)^2+1]$$

$$= \frac{(x_i-a)(b-x_i)}{24h^4}[(x_i-a)(b-x_i)+(b-a)^2+h^2]. \tag{5.1}$$

Proof: Lemma 5.1 gives the first equality. Then (2.1) and (2.2) give the second equality.

We now turn back to the error equation (3.4) and write it in the form

$$E = A^{-1}T$$

where the components of T are given by (3.1) or (3.2).

Clearly, $|E| \leqq |A^{-1}| \cdot |T|$, whence (4.15) and (4.18) imply

$$|E| \leqq A_0^{-1}|T|.$$

Thus,

$$|e_i| \leqq \sum_{j=1}^{N} a_{ij}^*|t_j| \leqq \frac{h^6}{6} M_6 \sum_{j=1}^{N} a_{ij}^*, \text{ by (3.6)}.$$

Now, from Lemma 5.2 we finally obtain

$$|e_i| \leqq \frac{h^2 M_6}{144}(x_i-a)(b-x_i)[(x_i-a)(b-x_i)+(b-a)^2+h^2]$$

$$\leqq \frac{h^2 M_6}{144}(x_i-a)(b-x_i)[(x_i-a)(b-x_i)+\tfrac{5}{4}(b-a)^2] \tag{5.2}$$

and, hence,

$$\|E\| \leqq \frac{h^2(b-a)^4}{384} M_6, \tag{5.3}$$

since

$$(x-a)(b-x) \leqq \tfrac{1}{4}(b-a)^2.$$

Thus,

$$\|E\| \leqq Ch^2,$$

where C is a positive constant independent of h, and it follows that $\|E\| \to 0$ as $h \to 0$. Therefore the method defined by (2.6) for the numerical integration of the boundary value problem (1.2) is convergent.

We summarize the above results in Theorem 5.1.

Theorem 5.1. *Let* $y(x)$ *be the exact solution of the boundary value problem* (1.2) *and let* y_n $(n = 1, 2, \ldots, N)$ *be the exact solution of the system of linear equations* (2.6). *Let* E *be given by* (3.5).

If

$$0 \leqq f(x) \leqq 36/(b-a)^4 \text{ on } [a, b],$$

then

$$\|E\| = O(h^2)$$

satisfies (5.3) *neglecting all errors due to round-off.*

Remark: Notice that (5.2) gives bounds on e_i which are specific to the relative location of x_i on the interval $[a, b]$. For example, for $i < N/6$ the bound on $|e_i|$ from (5.2) is less than $\frac{14}{27}Ch^2$.

6. Two numerical illustrations

In this concluding section we first illustrate the technique of this paper by a numerical example of the form (1.1). We choose

$$l = 2\,,\ D = 1\,,\ k = 4\,,\ q = 1\,,$$

from which the differential equation in (1.1) becomes

$$w^{(4)} + 4w = 1\,. \tag{6.1}$$

The exact solution is

$$w(x) = 0.25[1 - 2(\sin 1 \sinh 1 \sin x \sinh x + \cos 1 \cosh 1 \cos x \cosh x)/(\cos 2 + \cosh 2)]\,.$$

The numerical calculations were made on a PDP-10 computer (at the University of Pittsburgh Computer Center) using double precision arithmetic in order to reduce the round-off error to a minimum. The experimental results are briefly summarized in Tables 1 and 2.

Table 1 lists the observed maximum error for various N when (6.1) is solved numerically. Table 2 displays at intervals of 1/8 the computed values of w_i and e_i when N is 127.

To broaden the scope of the application of our method, we now consider an example of the form (1.2), namely,

$$y^{(4)} + xy = -(8 + 7x + x^3)e^x \tag{6.2}$$

TABLE 1

N	h	$\|\|E\|\|$
3	1/2	0.5209×10^{-2}
7	1/4	0.1289×10^{-2}
15	1/8	0.3215×10^{-3}
31	1/16	0.8031×10^{-4}
63	1/32	0.2007×10^{-4}
127	1/64	0.5018×10^{-5}

TABLE 2 ($N = 127$)

$\pm x_i$	w_i	$e_i \times 10^6$
1.0	0.0	0.0
0.875	0.025169	0.984
0.750	0.049150	1.929
0.625	0.070955	2.798
0.500	0.089793	3.557
0.375	0.105046	4.179
0.250	0.116257	4.640
0.125	0.123107	4.923
0.0	0.125411	5.018

TABLE 3

N	h	$\|\|E\|\|$
3	1/4	0.7160×10^{-2}
7	1/8	0.1744×10^{-2}
15	1/16	0.4330×10^{-3}
31	1/32	0.1081×10^{-3}
63	1/64	0.2703×10^{-4}
127	1/128	0.6756×10^{-5}

TABLE 4 ($N=127$)

x_i	y_i	$-e_i \times 10^6$
0.0	0.0	0.0
0.125	0.123941	2.519
0.250	0.240759	4.670
0.375	0.341020	6.152
0.500	0.412187	6.750
0.625	0.437876	6.350
0.750	0.396942	4.961
0.875	0.262380	2.736
1.000	0.0	0.

subject to the boundary conditions

$$\begin{aligned} y(0) &= 0 \qquad y(1) = 0 \\ y''(0) &= 0 \qquad y''(1) = -4e\,. \end{aligned} \tag{6.3}$$

The analytical solution of (6.2) and (6.3) is given by

$$y(x) = x(1-x)e^x\,.$$

We summarize our experimental results in Tables 3 and 4. Table 4 corresponds to $N=127$.

It is further verified from Tables 1 and 3 that on reducing the step-size from h to $h/2$, the maximum observed error in absolute value is approximately reduced by 1/4. This establishes the fact that our numerical method is of order 2, as Theorem 5.1 asserts. Tables 2 and 4 demonstrate the behavior of e_i which was noted in (5.2) above.

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